Error analysis in the reconstruction of a convolution kernel in a semilinear parabolic problem with integral overdetermination

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Abstract

A semilinear parabolic problem of second order with an unknown solely time-dependent convolution kernel is considered. An additional given global measurement (a space integral of the solution) ensures the existence of a unique weak solution. The unknown kernel function can be approximated by a time-discrete numerical scheme based on Backward Euler's method (Rothe's method). In this contribution, an error analysis for the time discretization is performed of the existing numerical algorithm. Numerical experiments support the theoretically obtained results.

Keywords: parabolic IBVP, integral overdetermination, convolution kernel, reconstruction, error analysis

1. Introduction

In this contribution, the domain Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$, with $\partial\Omega = \Gamma$ and $\Theta = [0, T]$, $T > 0$, the time frame. The aim of this paper is to derive estimates for the distance between the discrete and continuous solution of a semilinear parabolic problem. The former is based on a time-discrete numerical scheme, described in [1], that approximates the solution of the following semilinear parabolic problem: determine the solution u and the convolution kernel $K(t)$ such that

$$\begin{cases} \partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + K(t)h(\mathbf{x}, t) + (K * u(\mathbf{x}))(t) = f(u(\mathbf{x}, t), \nabla u(\mathbf{x}, t)), & \text{in } \Omega \times \Theta, \\ -\nabla u(\mathbf{x}, t) \cdot \nu = g(\mathbf{x}, t), & \text{on } \Gamma \times \Theta, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \text{in } \Omega \end{cases} \tag{1}$$

when an additional global measurement

$$\int_\Omega u(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = m(t) \tag{2}$$

is satisfied. Note that the data functions $h : \Omega \times \Theta \to \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $g : \Gamma \times \Theta \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$ and $m : \Theta \to \mathbb{R}$ are known, and time-convolution is defined as

$$(K * u(\mathbf{x}))(t) = \int_0^t K(t - s)\, u(\mathbf{x}, s)\,\mathrm{d}s, \quad t \in \Theta.$$

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Regarding f, one can replace it with $f + \varphi$ where $\varphi : \Omega \times \Theta \to \mathbb{R}$ is sufficiently regular. Such type of problems arise in the theory of reactive contaminant transport. In [2] one considers the following differential equation

$$\partial_t C + \nabla \cdot (\mathbf{V} C) - \Delta C = \frac{-\rho_b}{n} \partial_t S \tag{3}$$

for the aqueous concentration C and sorbed concentration per unit mass of solid S with mass transformation rate in first order form of

$$\partial_t S = K_r (K_d C - S)$$

with desorption rate K_r and equilibrium distribution coefficient K_d. This can be formally solved as

$$S(t) = \mathrm{e}^{-K_r t} S(0) + K_r K_d \int_0^t \mathrm{e}^{-K_r (t-\xi)} C(\xi) \, \mathrm{d}\xi.$$

Therefore, (3) can be rewritten as problem (1) for $u = C$ with $K(t) = -\frac{\rho_b}{n} K_r^2 K_d \mathrm{e}^{-K_r t}$ and $h(t) = \frac{S(0)}{K_r K_d}$. For an overview in the literature of papers dealing with integral overdetermination one may refer to [3, 4, 5, 6, 7, 8, 9, 10, 11]. Denote by $(\cdot, \cdot)$ the standard inner product of $\mathrm{L}^2(\Omega)$ and $\|\cdot\|$ its induced norm. The variational formulation of problem (1) reads as:
find $\langle u(t), K(t) \rangle \in \mathrm{H}^1(\Omega) \times \mathbb{R}$ with $\partial_t u(t) \in \mathrm{L}^2(\Omega)$ such that for all $\phi \in \mathrm{H}^1(\Omega, \mathbb{R})$ it holds

$$(\partial_t u, \phi) + (\nabla u, \nabla \phi) + (g, \phi)_\Gamma + K(t)(h, \phi) + (K * u, \phi) = (f(u, \nabla u), \phi), \quad a.e. \ t \in \Theta, \tag{P}$$

and such that the global measurement (2) *is satisfied.*
If we set $\phi = 1$ in (P) we obtain together with $(u, 1) = m(t)$

$$m'(t) + (g, 1)_\Gamma + K(t)(h, 1) + K * m = (f(u, \nabla u), 1). \tag{MP}$$

In [1], the authors proved the following existence and uniqueness theorem for the inverse problem:

Theorem 1 (see [1])**.** *Suppose f is bounded and Lipschitz continuous in all variables, $g \in \mathrm{C}^1(\Theta, \mathrm{L}^2(\Gamma))$, $h \in \mathrm{C}^0(\Theta, \mathrm{H}^1(\Omega)) \cap \mathrm{C}^1(\Theta, \mathrm{L}^2(\Omega))$ and $\min_{t \in \Theta} |(h(t), 1)| \geq \omega > 0$, $m \in \mathrm{C}^2(\Theta, \mathbb{R})$ and $u_0 \in \mathrm{H}^2(\Omega)$. Then there exists a unique couple solutions $\langle u, K \rangle$ to* (P)-(MP)*, where $u \in \mathrm{C}(\Theta, \mathrm{H}^1(\Omega))$, $\partial_t u \in \mathrm{L}^\infty(\Theta, \mathrm{L}^2(\Omega))$ and $K \in \mathrm{C}(\Theta)$, $K' \in \mathrm{L}^2(\Theta)$.*

The outline of this paper is as follows. In Section 2, a time-discrete scheme to approximate the solution to problem (1)-(2) is described. The corresponding error estimates are derived in Section 3. Finally, some numerical experiments are developed in Section 4.

2. Numerical scheme

2.1. Discretization

We apply the Rothe method [12, 13]. Consider an equidistant time-partitioning of the time frame Θ with a step $\tau = T/n < 1$, for any $n \in \mathbb{N}$. We use the notation $t_i = i\tau$ and for any function z we write

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

At time t_i we infer from (P) the backward Euler scheme

$$(\delta u_i, \phi) + (\nabla u_i, \nabla\phi) + (g_i, \phi)_\Gamma + K_i(h_i, \phi) + \sum_{k=1}^{i} (K_k u_{i-k}\tau, \phi) = (f_{i-1}, \phi) \tag{DPi}$$

where $f_i = f(u_i, \nabla u_i)$. This is conveniently written as $B(u_i, \phi) = F_i(\phi)$ with

$$B(u_i, \phi) = \frac{1}{\tau}(u_i, \phi) + (\nabla u_i, \nabla\phi), \quad F_i(\phi) = (f_{i-1}, \phi) - (g_i, \phi)_\Gamma - K_i(h_i, \phi) - \sum_{k=1}^{i} (K_k u_{i-k}\tau, \phi) + \frac{1}{\tau}(u_{i-1}, \phi).$$

Analogously, we obtain from (MP)

$$m_i' + (g_i, 1)_\Gamma + K_i(h_i, 1) + \sum_{k=1}^{i} K_k m_{i-k}\tau = (f_{i-1}, 1). \tag{DMPi}$$

Using (DPi) and (DMPi) the numerical algorithm is as follows:

Algorithm: numerical scheme in pseudo code

input : $T > 0$, $n \in \mathbb{N}$ and functions f, g, h, m and u_0
output: kernel K and solution u at discrete time steps

1 $\tau \leftarrow T/n$;
2 $\theta \leftarrow [0 : \tau : T]$;
3 $\mathsf{K} \leftarrow \mathtt{zeros}(n+1)$;
4 $\mathsf{u} \leftarrow \mathtt{eval}(u_0, \theta)$;
5 $\mathsf{K}[0] \leftarrow \dfrac{1}{(h_0, 1)}\Big((f(u_0, \nabla u_0), 1) - m_0' - (g_0, 1)_\Gamma\Big)$;
6 **for** $i = 1$ **to** n **do**
7 $\quad \mathsf{K}[i] \leftarrow \dfrac{1}{(h_i, 1) + m_0\tau}\left((f_{i-1}, 1) - (g_i, 1)_\Gamma - \sum_{k=1}^{i-1} K_k m_{i-k}\tau - m_i'\right)$;
8 $\quad \mathsf{u}[i] \leftarrow \mathtt{solveEP}(B(u_i, \phi) = F_i(\phi))$;

Note that on Line 7 one needs $(h_i, 1) + m_0\tau \neq 0$. However $(h_i, 1)$ is never null so τ can be chosen accordingly. Based on the output of the above algorithm, we introduce the following piecewise linear function in time

$$u_n : \Theta \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 0 \leq i \leq n,$$

and a step function

$$\bar{u}_n : \Theta \to \mathrm{L}^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0 \\ u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 0 \leq i \leq n.$$

Similarly, we define $\bar{K}_n$, $\bar{h}_n$, $\bar{g}_n$, $\bar{m}_n$ and $\overline{m'}_n$. Using Rothe's functions, we can write (DPi) and (DMPi) on the whole time frame as

$$(\partial_t u_n, \phi) + (\nabla \bar{u}_n, \nabla \phi) + (\bar{g}_n, \phi)_\Gamma + \bar{K}_n(\bar{h}_n, \phi) + \sum_{k=1}^{\lfloor t \rfloor_\tau} (\bar{K}_n(t_k)\bar{u}_n(t-t_k)\tau, \phi) = (f(\bar{u}_n(t-\tau), \nabla \bar{u}_n(t-\tau)), \phi). \quad \text{(DP)}$$

where $\lfloor t \rfloor_\tau = i$ when $t \in (t_{i-1}, t_i]$, and

$$\overline{m'}_n + (\bar{g}_n, 1)_\Gamma + \bar{K}_n(\bar{h}_n, 1) + \sum_{k=1}^{\lfloor t \rfloor_\tau} \bar{K}_n(t_k)\bar{m}_n(t-t_k)\tau = (f(\bar{u}_n(t-\tau), \nabla \bar{u}_n(t-\tau)), 1). \quad \text{(DMP)}$$

2.2. Useful inequalities

Two frequently used estimates for the convolution term are:

Lemma 1. *Set* $I = [0, \eta]$, $\eta > 0$. *Suppose* $\kappa \in \mathrm{L}^2(I)$ *and* $\upsilon \in \mathrm{L}^1(I, \mathrm{L}^2(\Omega))$ *then it holds that*

$$\|\kappa * \upsilon\|^2 \le \kappa^2 * \|\upsilon\|^2, \quad (*)$$

$$\int_0^\eta \|\kappa * \upsilon\|^2 \, \mathrm{d}t \le \int_0^\eta |\kappa|^2 \, \mathrm{d}t \int_0^\eta \|\upsilon\|^2 \, \mathrm{d}t. \quad (**)$$

Proof. From Jensen's inequality it follows that

$$\begin{aligned} \|\kappa * \upsilon\|^2 = \int_\Omega (\kappa * \upsilon)^2 \, \mathrm{d}\mathbf{x} &= \int_\Omega \left(\int_0^t \kappa(s)\upsilon(\mathbf{x}, t-s) \, \mathrm{d}s \right)^2 \mathrm{d}\mathbf{x} \le \int_\Omega \int_0^t \kappa(s)^2 \upsilon(\mathbf{x}, t-s)^2 \, \mathrm{d}s \, \mathrm{d}\mathbf{x} \\ &= \int_0^t \kappa(s)^2 \|\upsilon(t-s)\|^2 \, \mathrm{d}s = \kappa^2 * \|\upsilon\|^2, \end{aligned}$$

from which we obtain

$$\left\| \|\kappa * \upsilon\|^2 \right\|_{\mathrm{L}^1(I)} \le \left\| \kappa^2 * \|\upsilon\|^2 \right\|_{\mathrm{L}^1(I)} \le \left\| \kappa^2 \right\|_{\mathrm{L}^1(I)} \left\| \|\upsilon\|^2 \right\|_{\mathrm{L}^1(I)}$$

by Young's inequality for convolutions. □

3. Error analysis

In this section we will derive error bounds on the kernel K and solution u which results in reps. convergence rates for the proposed numerical scheme. Based on applying δ to (DPi) one proves:

Lemma 2. *Let the conditions of Theorem 1 be fulfilled. If* $\langle u_i, K_i \rangle$ *is a weak solution of* (1) *at time step i, then there exists* $C > 0$ *such that for each* $t_j \in \Theta$ *one has* $\left\| \delta u_j \right\|^2 \le C$ *and* $\left\| \nabla \delta u_j \right\|^2 \le C$.

Proof. The fact that $\left\| \delta u_j \right\|^2 \le C$ is explicitly proved in [1]. The inequality $\left\| \nabla \delta u_j \right\|^2 \le C$ is completely similar and therefore we omit it. □

Lemma 3. *Let the conditions of Theorem 1 be fulfilled. If* $\langle u_i, K_i \rangle$ *is a weak solution of* (1) *at time step i, then there exists* $C > 0$ *such that for each* $t_j \in \Theta$ *one has* $\left| \delta K_j \right| \le C$.

Proof. The fact that $u_0 \in \mathrm{H}^2(\Omega)$ implies that the PDE from (1) is fulfilled at $t = 0$, i.e. one can define the initial value for $\partial_t u$ in the following way

$$\partial_t u(0) := f(u_0, \nabla u_0) + \Delta u_0 - K(0)h(0) \in \mathrm{L}^2(\Omega).$$

Applying measurement to this equation gives

$$m'_0 + (g_0, 1)_\Gamma + K_0(h_0, 1) = (f_0, 1). \tag{DMP0}$$

We would like to apply the δ-operator to (DMPi). Using the rule $\delta(a_i b_i) = \delta a_i\, b_i + a_{i-1}\, \delta b_i$ we get for $i \geq 2$

$$\delta m'_i + (\delta g_i, 1)_\Gamma + \delta K_i\, (h_i, 1) + K_{i-1}(\delta h_i, 1) + K_i m_0 + \sum_{k=1}^{i-1} K_k \delta m_{i-k} \tau = (\delta f_{i-1}, 1).$$

Thus for $i \geq 2$ it holds

$$\begin{aligned} |\delta K_i|\, |(h_i, 1)| &\leq \left|\delta m'_i\right| + |(\delta g_i, 1)_\Gamma| + |K_{i-1}(\delta h_i, 1)| + |K_i m_0| + \sum_{k=1}^{i-1} |K_k \delta m_{i-k}|\, \tau + |(\delta f_{i-1}, 1)| \\ &\leq C + C\left(\|\delta u_{i-1}\| + \|\delta \nabla u_{i-1}\|\right). \end{aligned}$$

Further, we subtract (DMP0) from (DMPi) for $i = 1$ to get

$$\delta m'_1 + (\delta g_1, 1)_\Gamma + \delta K_1\, (h_1, 1) + K_0(\delta h_1, 1) + K_1 m_0 = 0$$

and we estimate

$$|\delta K_1|\ \ |(h_1, 1)| \leq |K_1 m_0| + |K_0(\delta h_1, 1)| + |(\delta g_1, 1)_\Gamma| + \left|\delta m'_1\right|.$$

The proof is completed by Lemma 2 and $|(h_i, 1)| \geq \omega > 0$. □

Theorem 2 (Error on convolution kernel)**.** *Let the conditions of Theorem 1 be fulfilled. Then, there exists a positive constant C, independent of the time step τ, such that*

$$\max_{t \in [0,T]} \left|\bar{K}_n(t) - K(t)\right| \leq C\tau.$$

Proof. Note that K_n converges uniformly to K and each K_n is (piecewise) Lipschitz as $\partial_t K_n = \delta K_i$ is bounded, see Lemma 3. By the Arzelà-Ascoli theorem [14, Theorem 11.28] it follows that K is (piecewise) Lipschitz continuous. From this we obtain

$$\max_{t \in [0,T]} \left|\bar{K}_n(t) - K(t)\right| = \max_{1 \leq i \leq n} \max_{t \in [t_{i-1}, t_i]} |K(t_i) - K(t)| \leq \max_{1 \leq i \leq n} \max_{t \in [t_{i-1}, t_i]} C_i\, |t_i - t| \leq C\tau,$$

which concludes the proof. □

A direct consequence is

Corollary 1. *Let the conditions of Theorem 1 be fulfilled. Then, there exists a positive constant C, independent of the time step τ, such that*

$$\int_0^T \left|\bar{K}_n(t) - K(t)\right|^2 \mathrm{d}t \leq C\tau^2.$$

Theorem 3 (Error on solution)**.** *Let the conditions of Theorem 1 be fulfilled. Then, there exists a positive constant C, independent of the time step τ, such that*

$$\max_{t\in[0,T]} \|u_n(t) - u(t)\|^2 + \int_0^T \|\nabla u_n(t) - \nabla u(t)\|^2 \,\mathrm{d}t \le C\tau^2.$$

Proof. We subtract (P) from (DP)

$$\begin{aligned}(\partial_t(u_n - u), \phi) + (\nabla(\bar{u}_n - u), \nabla\phi) + (\bar{g}_n - g, \phi)_\Gamma + \bar{K}_n(t)(\bar{h}_n, \phi) - K(t)(h, \phi) \\ + (\bar{K}_n * \bar{u}_n - K * u, \phi) = (f(\bar{u}_n(t-\tau), \nabla\bar{u}_n(t-\tau)) - f(u, \nabla u), \phi).\end{aligned} \tag{4}$$

We adopt the following notations

$$\begin{aligned}e_{\bar{K}}(t) &= \bar{K}_n(t) - K(t)\\ e_u(\mathbf{x}, t) &= u_n(\mathbf{x}, t) - u(\mathbf{x}, t)\\ e_{\bar{g}}(\mathbf{x}, t) &= \bar{g}_n(\mathbf{x}, t) - g(\mathbf{x}, t)\\ e_{\bar{h}}(\mathbf{x}, t) &= \bar{h}_n(\mathbf{x}, t) - h(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times \Theta.\end{aligned}$$

Therefore (4) is rewritten as

$$\begin{aligned}(\partial_t e_u, \phi) + (\nabla e_u, \nabla\phi) + (\nabla(\bar{u}_n - u_n), \nabla\phi) + (e_{\bar{g}}, \phi)_\Gamma + e_{\bar{K}}(\bar{h}_n, \phi) + K(t)(e_{\bar{h}}, \phi) \\ + (\bar{K}_n * (\bar{u}_n - u_n), \phi) + (\bar{K}_n * e_u, \phi) + (e_{\bar{K}} * u, \phi) = (f(\bar{u}_n(t-\tau), \nabla\bar{u}_n(t-\tau)) - f(u, \nabla u), \phi).\end{aligned}$$

Note that

$$\|\bar{u}_n(t-\tau) - u(t)\|^2 \le \|\bar{u}_n(t-\tau) - \bar{u}_n(t)\|^2 + \|\bar{u}_n(t) - u(t)\|^2 = \tau^2 \|\partial_t u_n(t)\|^2 + \|\bar{u}_n(t) - u(t)\|^2, \quad \text{a.e. } t \in \Theta$$

and likewise

$$\|\nabla\bar{u}_n(t-\tau) - \nabla u(t)\|^2 \le \tau^2 \|\partial_t \nabla u_n(t)\|^2 + \|\nabla\bar{u}_n(t) - \nabla u(t)\|^2, \quad \text{a.e. } t \in \Theta.$$

This combined with the Lipschitz continuity of f results in

$$\begin{aligned}&|(f(\bar{u}_n(t-\tau), \nabla\bar{u}_n(t-\tau)) - f(u, \nabla u), \phi)|\\ &\quad\le C\left(\|\bar{u}_n(t-\tau) - u(t)\|^2 + \|\nabla\bar{u}_n(t-\tau) - \nabla u(t)\|^2\right)^{1/2} \|\phi\|\\ &\quad\le C\left(\tau^2 \|\partial_t u_n(t)\|^2 + \|\bar{u}_n(t) - u(t)\|^2 + \tau^2 \|\partial_t \nabla u_n(t)\|^2 + \|\nabla\bar{u}_n(t) - \nabla u(t)\|^2\right)^{1/2} \|\phi\|\\ &\quad\le \epsilon\tau^2 \|\partial_t u_n(t)\|^2_{\mathrm{H}^1(\Omega)} + \epsilon \|\bar{u}_n - u_n\|^2_{\mathrm{H}^1(\Omega)} + \epsilon \|e_u\|^2_{\mathrm{H}^1(\Omega)} + C_\epsilon \|\phi\|^2.\end{aligned}$$

By the trace theorem, we have that

$$\left|(e_{\bar{g}}, \phi)_\Gamma\right| \le C \left\|e_{\bar{g}}\right\|_{\mathrm{L}^2(\Gamma)} \|\phi\|_{\mathrm{H}^1(\Omega)}$$

and further one has by Cauchy-Schwarz inequality

$$\begin{aligned}
\left|e_{\bar{K}}(\bar{h}_n,\phi)\right| &\leq \left|e_{\bar{K}}\right| \left\|\bar{h}_n\right\| \|\phi\| \leq C\left|e_{\bar{K}}\right| \|\phi\|, \\
\left|K(t)(e_{\bar{h}},\phi)\right| &\leq |K| \left\|e_{\bar{h}}\right\| \|\phi\| \leq C\left\|e_{\bar{h}}\right\| \|\phi\|, \\
\left|(\bar{K}_n * e_u,\phi)\right| &\overset{(*)}{\leq} \frac{1}{2}\bar{K}_n^2 * \|e_u\|^2 + \frac{1}{2}\|\phi\|^2, \\
\left|(e_{\bar{K}} * u,\phi)\right| &\overset{(*)}{\leq} \frac{1}{2}e_{\bar{K}}^2 * \|u\|^2 + \frac{1}{2}\|\phi\|^2.
\end{aligned}$$

When we put $\phi = e_u$ in (4) and integrate over $(0,\eta) \subset \Theta$, we obtain

$$\begin{aligned}
\frac{1}{2}\|e_u(\eta)\|^2 + \int_0^\eta \|\nabla e_u\|^2 \leq \epsilon\tau^2 \int_0^\eta \|\partial_t u_n(t)\|^2_{\mathrm{H}^1(\Omega)} + \epsilon \int_0^\eta \|\bar{u}_n - u_n\|^2_{\mathrm{H}^1(\Omega)} + \epsilon \int_0^\eta \|e_u\|^2_{\mathrm{H}^1(\Omega)} + C_\epsilon \int_0^\eta \|e_u\|^2 \\
+ \int_0^\eta (\nabla(\bar{u}_n - u_n), \nabla e_u) + C\int_0^\eta \left\|e_{\bar{g}}\right\|_{\mathrm{L}^2(\Gamma)} \|e_u\|_{\mathrm{H}^1(\Omega)} + C\int_0^\eta \left|e_{\bar{K}}\right|^2 + C\int_0^\eta \left\|e_{\bar{h}}\right\|^2 \\
+ \frac{1}{2}\int_0^\eta \bar{K}_n^2 * \|e_u\|^2 + \frac{1}{2}\int_0^\eta e_{\bar{K}}^2 * \|\bar{u}_n - u_n\|^2 + \frac{1}{2}\int_0^\eta e_{\bar{K}}^2 * \|u\|^2, \quad (5)
\end{aligned}$$

where we used $\|e_u(0)\| = 0$. Similarly to Corollary 1 we have from the Lipschitz continuity of h and g that

$$\begin{aligned}
\int_0^\eta \left\|e_{\bar{g}}\right\|^2_{\mathrm{L}^2(\Gamma)} \mathrm{d}t &= \int_0^\eta \|\bar{g}_n(t) - g(t)\|^2_{\mathrm{L}^2(\Gamma)} \mathrm{d}t \leq C\tau^2, \\
\int_0^\eta \left\|e_{\bar{h}}\right\|^2 \mathrm{d}t &= \int_0^\eta \left\|\bar{h}_n(t) - h(t)\right\|^2 \mathrm{d}t \leq C\tau^2.
\end{aligned}$$

Now, inequality (5) simplifies to

$$\begin{aligned}
\frac{1}{2}\|e_u\|^2 + \frac{1}{2}\int_0^\eta \|\nabla e_u\|^2 \leq C\tau^2 + \epsilon\int_0^\eta \|\bar{u}_n - u_n\|^2_{\mathrm{H}^1(\Omega)} + \epsilon\int_0^\eta \|\nabla e_u\|^2 + \frac{1}{2}\int_0^\eta \|\nabla(\bar{u}_n - u_n)\|^2 + C_\epsilon \int_0^\eta \|e_u\|^2 \\
+ C\int_0^\eta \left|e_{\bar{K}}\right|^2 + \frac{1}{2}\int_0^\eta \bar{K}_n^2 * \|e_u\|^2 + \frac{1}{2}\int_0^\eta e_{\bar{K}}^2 * \|\bar{u}_n - u_n\|^2 + \frac{1}{2}\int_0^\eta e_{\bar{K}}^2 * \|u\|^2
\end{aligned}$$

as $\|\partial_t u_n(t)\|^2_{\mathrm{H}^1(\Omega)}$ is $\mathrm{L}^1(\Theta)$-bounded by [1] and $\sup_{t\in\Theta} \|e_u\|_{\mathrm{H}^1(\Omega)} \leq C$ as $u \in \mathrm{L}^\infty(\Theta, \mathrm{H}^1(\Omega))$. Together with $(**)$ this results in

$$\begin{aligned}
\|e_u\|^2 + (1-2\epsilon)\int_0^\eta \|\nabla e_u\|^2 \leq C\tau^2 + 2\epsilon\int_0^\eta \|\bar{u}_n - u_n\|^2 + \int_0^\eta \|\nabla(\bar{u}_n - u_n)\|^2 + C_\epsilon\int_0^\eta \|e_u\|^2 \\
+ C\int_0^\eta \left|e_{\bar{K}}\right|^2 + \int_0^\eta e_{\bar{K}}^2 \int_0^\eta \|\bar{u}_n - u_n\|^2 + \int_0^\eta e_{\bar{K}}^2 \int_0^\eta \|u\|^2
\end{aligned}$$

as K_n is bounded. From Corollary 1, we infer

$$\|e_u\|^2 + (1-\epsilon)\int_0^\eta \|\nabla e_u\|^2 \leq C\tau^2 + (C\tau^2 + \epsilon)\int_0^\eta \|\bar{u}_n - u_n\|^2 + \int_0^\eta \|\nabla(\bar{u}_n - u_n)\|^2 + C_\epsilon\int_0^\eta \|e_u\|^2$$

again as $u \in \mathrm{C}(\Theta, \mathrm{L}^2(\Omega))$. As $\|\bar{u}_n - u_n\| = |t_i - t|\,\|\delta u_i\| \leq \tau\,\|\partial_t u_n(t)\|$ and $\|\nabla(\bar{u}_n - u_n)\| = |t_i - t|\,\|\nabla \delta u_i\| \leq \tau\,\|\nabla \partial_t u_n(t)\|$ when $t \in (t_{i-1}, t_i]$, we have

$$\|e_u\|^2 + (1-\epsilon)\int_0^\eta \|\nabla e_u\|^2 \leq C\tau^2 + C_\epsilon \int_0^\eta \|e_u\|^2 + \epsilon\tau^2 \leq C\tau^2 + C_\epsilon \int_0^\eta \|e_u\|^2$$

again as $\|\partial_t u_n(t)\|^2_{\mathrm{H}^1(\Omega)}$ is $\mathrm{L}^1(\Theta)$-bounded. Fixing a suitable $\epsilon > 0$ the proof is concluded by Grönwall's lemma as $\eta \leq T$ was chosen arbitrarily. □

4. Numerical Experiments

The aim of the simulations is to demonstrated the convergence of the numerical procedure proposed in Section 2 and to verify the theoretically established error estimates in Theorem 2 and 3. The finite element library DOLFIN [15, 16] from the collaborative FEniCS project [17] is used for the implementation.

In each experiment, it is assumed that $\Omega = [0,1] \subset \mathbb{R}$. The forward problems in the procedure are discretized in time accordingly to the backward Euler method. The number of time discretization interval is chosen to be $n = 2^j$, $j = 5, \ldots, 9$, such that the time step τ for the equidistant time partitioning equals respectively $2^{-j}T$, $j = 5, \ldots, 9$. At each time-step, the resulting elliptic problems (see Line 8 in the Algorithm) are solved numerically by the finite element method (FEM) using first order (P1-FEM) Lagrange polynomials for the space discretization. A fixed uniform mesh of 50 intervals is used. The error between the numerical and exact solution for the several values of the timestep τ is computed. This errors are respectively denoted by

$$E_K(\tau) = \max_{t\in[0,T]} |\bar{K}_n(t) - K_{\mathrm{ex}}(t)| \approx \max_{0\leqslant i\leqslant n} |\bar{K}_n(t_i) - K_{\mathrm{ex}}(t_i)|$$

and

$$E_u(\tau) = \max_{t\in[0,T]} \|u_n(t) - u_{\mathrm{ex}}(t)\|^2 \approx \max_{0\leqslant i\leqslant n} \|u_n(t_i) - u_{\mathrm{ex}}(t_i)\|^2 .$$

We perform three experiments. The first two have a homogeneous Neumann boundary condition. Note that the RHS of problem (1) is rewritten as $f(u(x,t), \nabla u(x,t)) + F(x,t)$ in the numerical experiments.

Experiment 1

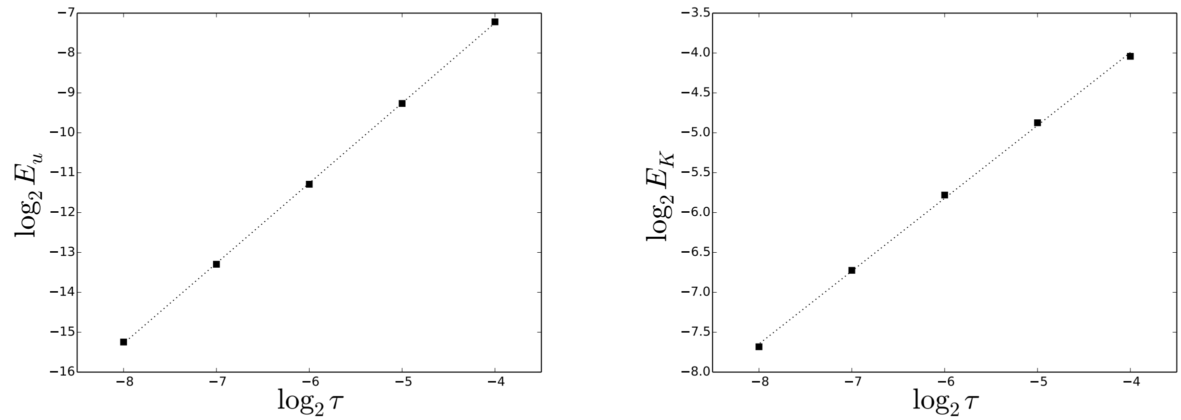
The data functions in the first experiment are prescribed as follows

$$\begin{aligned}
T &= 2, \quad f(r,s) = \sqrt{r^2 + \pi}, \\
F(x,t) &= \cos(\pi x)\left(t^2 + \pi^2 t^2 + t + \pi^2 t + 3 + \pi^2\right) + t^2 + t + 3 - 2\,\mathrm{e}^{-t}\cos(\pi x) + \mathrm{e}^{-t}(t + x - 2) \\
&\quad - \sqrt{\left(t^2 + t + 1\right)^2 (\cos(\pi x) + 1)^2 + \pi}, \\
h(x,t) &= x + t, \quad u_0(x) = 1 + \cos(\pi x), \\
g(x,t) &= 0, \quad m(t) = t^2 + t + 1,
\end{aligned}$$

such that the exact solution is given by

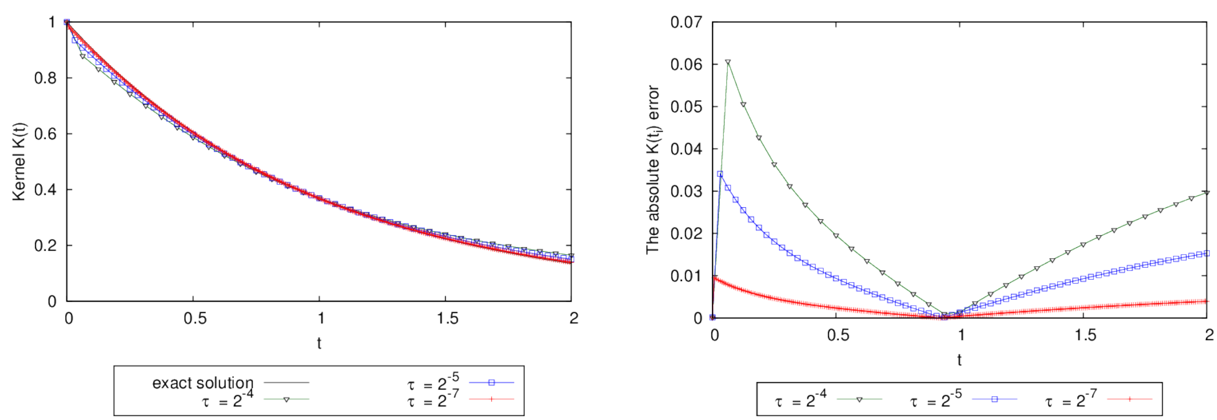
$$u_{\mathrm{ex}}(x,t) = \left(t^2 + t + 1\right)(\cos(\pi x) + 1) \quad \& \quad K_{\mathrm{ex}}(t) = \mathrm{e}^{-t}. \tag{6}$$

Note that $(h, 1) = \frac{1}{2} + t \neq 0, \forall t \in \Theta$. As mentioned before, the errors $E_K(\tau)$ and $E_u(\tau)$ are derived for $\tau = 2^{-j+1}$, $5 \leqslant j \leqslant 9$. They are depicted in Figure 1, where the errors $\log_2 E_K$ and $\log_2 E_u$ are plotted as a function of $\log_2 \tau$. The linear regression lines through all the data points are given by $\log_2 E_K = 0.9132 \log_2 \tau - 0.3412$ and $\log_2 E_u = 2.0086 \log_2 \tau + 0.7784$ respectively. This is in accordance with the predicted convergence rates $E_K \approx O(\tau)$ and $E_u \approx O(\tau^2)$ in Theorem 2 and 3 respectively. The exact kernel K_{ex} is compared with the numerical solution for $\tau = 2^{-4}, 2^{-5}$ and $\tau = 2^{-7}$ in Figure 2(a). The absolute $K(t_i)$-error, i.e. $|\bar{K}_n(t_i) - K_{\text{ex}}(t_i)|$ for $i = 0, \ldots, n$, is given in Figure 2(b) for $\tau = 2^{-4}, 2^{-5}$ and $\tau = 2^{-7}$.



(a) error on u (b) error on K

Figure 1: Convergence rates for Experiment 1 on logarithmic scale.



(a) exact solution (6) and numerical solution (b) absolute $K(t_i)$-error

Figure 2: Kernel reconstruction in Experiment 1.

Experiment 2

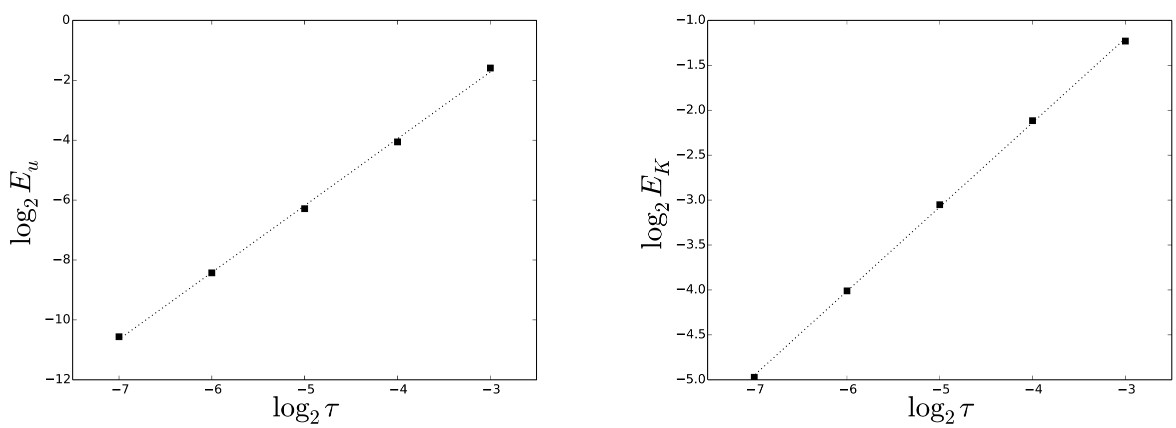
In the second experiment the data is chosen such that the unknown kernel is sinusoidal, i.e.

$$
\begin{aligned}
T &= 4, \quad f(r,s) = \sqrt{r^2+s^2+\pi}, \\
F(x,t) &= \frac{1}{2\pi^3}\Big[\cos(\pi x)\left(2\pi^5t^2+2\pi^5t+2\pi^5+4\pi^3t+\pi^2t+2\pi^2+2\pi^3-1\right)+2\sin\left(2\pi t+\pi^2t^2\right)\pi^3t \\
&\quad +2\sin(2\pi t)\pi^3x+(\cos(\pi t))^2\left(2\pi^2\cos(\pi x)+\cos(\pi x)-2\pi^2+1\right) \\
&\quad -\cos(\pi x)\pi\sin(\pi t)\cos(\pi t)-\pi\sin(\pi t)\cos(\pi t)+4\pi^3t+\pi^2t^2+2\pi^3+\pi^2t+2\pi^2-1\Big] \\
&\quad -\sqrt{(t^2+t+1)^2(\cos(\pi x)+1)^2+(t^2+t+1)^2\pi^2\sin(\pi x)^2+\pi}, \\
h(x,t) &= x+t, \quad u_0(x) = 1+\cos(\pi x), \\
g(x,t) &= 0, \quad m(t) = t^2+t+1,
\end{aligned}
$$

with exact solution

$$
u_{\mathrm{ex}}(x,t) = \left(t^2+t+1\right)(\cos(\pi x)+1) \quad \& \quad K_{\mathrm{ex}}(t) = \sin(2\pi t). \tag{7}
$$

The results of the numerical experiment are depicted in Figures 3–4. Now, the linear regression lines are given by $\log_2 E_K = 0.9378\log_2\tau + 1.6130$ and $\log_2 E_u = 2.2313\log_2\tau + 4.9715$. Again, a good numerical approximation of the unknown kernel can be obtained if the time step is sufficiently small.

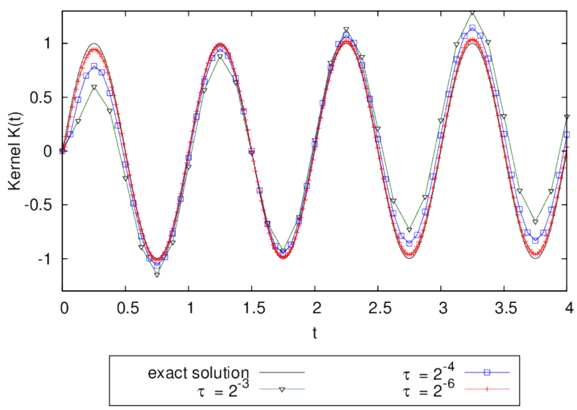


(a) error on u (b) error on K

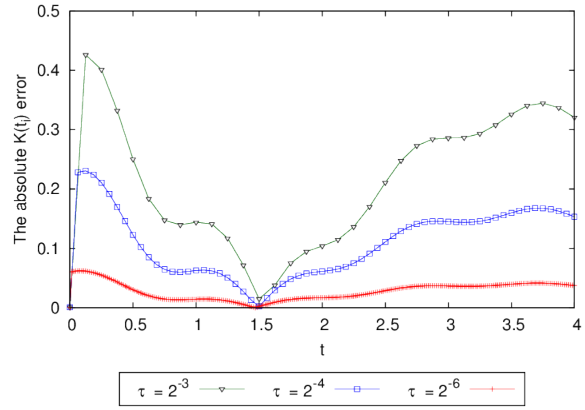
Figure 3: Convergence rates for Experiment 2 on logarithmic scale.

Experiment 3

The exact solution in the third experiment is selected such that the boundary condition is non-homogeneous, i.e. $g \not\equiv 0$. Consider the following data functions



(a) exact solution (7) and numerical solution



(b) absolute $K(t_i)$-error

Figure 4: Kernel reconstruction in Experiment 2.

$$
\begin{aligned}
T &= 4, \quad f(r,s) = \cos\left(r^2\right), \\
F(x,t) &= \frac{1}{(xt+1)^2\,(t+1)^2\,(xt+x+1)}\Big[t^4 + 2\,t^3 + t^2 + 2\,x^2 + t^3x^4 + 2\,t^4x^3 + t^5x + 4\,t^3x^3 + t^2x^4 + 3\,t^4x \\
&\quad + 5\,t^3x^2 + 6\,t^2x^3 + 3\,t^3x + 7\,t^2x^2 + 3\,tx^3 + 5\,t^2x + 7\,tx^2 + 2\,x + 3\,xt + t \\
&\quad - \ln(t+1)\left(x + t^2x + 2tx^2 + 2tx + t^4x^3 + 2t^3x^3 + 2t^3x^2 + t^2x^3 + 4t^2x^2\right) \\
&\quad + \ln(xt+1)\left(t^5x^3 + 2t^4x^3 + 3t^4x^2 + t^3x^3 + 6t^3x^2 + 3t^3x + 3t^2x^2 + 6t^2x + 3xt + 1 + t^2 + 2t\right)\Big] \\
&\quad - \cos\left(\ln(xt+1)^2\right), \\
h(x,t) &= x+t, \quad g(x,t) = \begin{cases} t & x=0 \\ \frac{t}{t+1} & x=1 \end{cases}, \quad u_0(x) = 0, \\
m(t) &= \begin{cases} 0 & t=0 \\ \frac{\ln(t+1)t+\ln(t+1)-t}{t} & t\in(0,1] \end{cases}, \quad m'(t) = \begin{cases} \frac{1}{2} & t=0 \\ \frac{t-\ln(t+1)}{t^2} & t\in(0,1] \end{cases},
\end{aligned}
$$

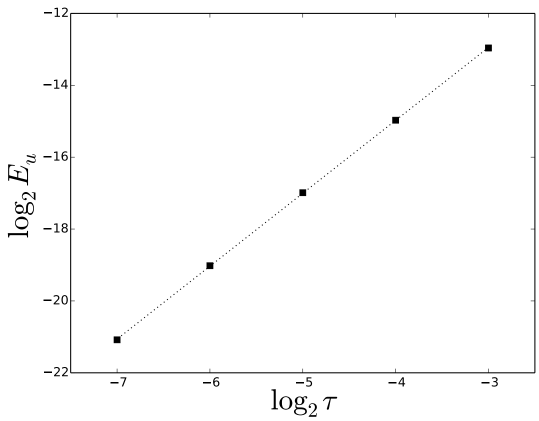
such that

$$
u_{\text{ex}}(x,t) = \ln(xt+1) \quad \& \quad K_{\text{ex}}(t) = \frac{1}{(1+t)^2}. \tag{8}
$$

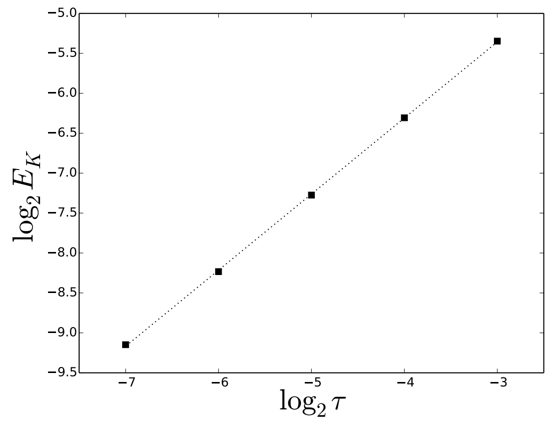
The results are in accordance with the preceding numerical experiments, see Figure 5–6. The linear regression lines are given by $\log_2 E_K = 0.9528\log_2\tau - 2.4981$ and $\log_2 E_u = 2.0286\log_2\tau - 6.8616$.

5. Conclusion

A semilinear parabolic problem of second order with an unknown solely time-dependent convolution kernel is considered. A numerical scheme based on Backward Euler's method together with a time-discrete convolution is presented in oder to reconstruct the unknown convolution kernel based on an integral overdetermination. It is proved that the convergence is of first order in time: $\max_{t\in[0,T]}|\bar{K}_n(t) - K_{\text{ex}}(t)| \approx \mathcal{O}(\tau)$

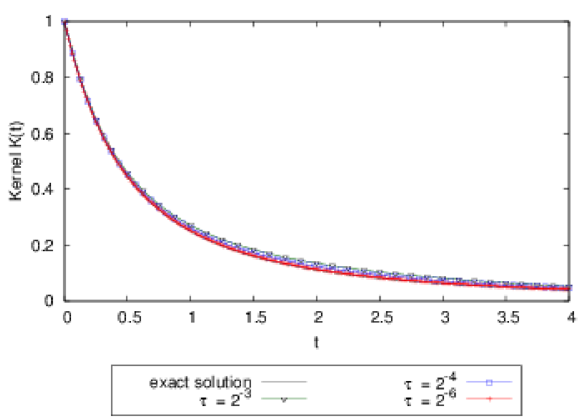


(a) error on u

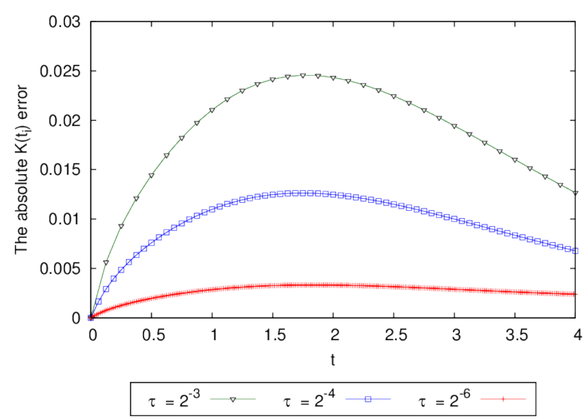


(b) error on K

Figure 5: Convergence rates for Experiment 3 on logarithmic scale.



(a) exact solution (8) and numerical solution



(b) absolute $K(t_i)$-error

Figure 6: Kernel reconstruction in Experiment 3.

and $\max_{t\in[0,T]} \|u_n(t) - u_{\rm ex}(t)\| \approx O(\tau)$. Three numerical experiments (homogeneous and non-homogeneous Neumann boundary condition) are conducted, which all support the theoretically obtained results.

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